2. $P=\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}, D=\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute

$P^{-1}=\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, D^4=\begin{bmatrix} 1 & 0 \\ 0 & 1/16 \end{bmatrix}$, and $A^4=\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/16 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}=\frac{1}{16}\begin{bmatrix} 151 & 90 \\ -225 & -134 \end{bmatrix}$

4. $A^k = PD^kP^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 1^k \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 4-3\cdot 2^k & 12\cdot 2^k-12 \\ 1-2^k & 4\cdot 2^k-3 \end{bmatrix}.$

6. As in Exercise 5, inspection of the factorization gives:

$$\lambda = 4: \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}; \lambda = 5: \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

12. The eigenvalues of A are given to be 2 and 8.

For $\lambda = 8$: $A-8I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}$, and row reducing $[A-8I \quad \mathbf{0}]$ yields $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$: $A-2I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$, and row reducing $[A-2I \quad \mathbf{0}]$ yields $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

18. An eigenvalue of A is given to be 5; an eigenvector $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is also given. To find the eigenvalue corresponding to $\mathbf{v}_1$, compute $A\mathbf{v}_1 = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 6 \end{bmatrix} = -3\mathbf{v}_1$. Thus the eigenvalue in question is -3.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -12 & -16 & 4 \\ 6 & 8 & -2 \\ 12 & 16 & -4 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-5I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 4/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution is $x_2\begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$, and a nice basis for the eigenspace is

$$\{\mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}.$$

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 1 \\ 1 & 3 & 0 \\ 2 & 0 & 3 \end{bmatrix}$. Then set $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively. Note that this answer differs from the text. There, $P = \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1 \end{bmatrix}$ and the entries in D are rearranged to match the new order of the eigenvectors. According to the Diagonalization Theorem, both answers are correct.

26. Yes, if the third eigenspace is only one-dimensional. In this case, the sum of the dimensions of the eigenspaces will be six, whereas the matrix is 7×7. See Theorem 7(b). An argument similar to that for Exercise 24 can also be given.

30. A nonzero multiple of an eigenvector is another eigenvector. To produce P_2, simply multiply one or both columns of P by a nonzero scalar unequal to 1.

32. Any 2×2 matrix with two distinct eigenvalues is diagonalizable, by Theorem 6. If one of those eigenvalues is zero, then the matrix will not be invertible. Any matrix of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ has the desired properties when a and b are nonzero. The number a must be nonzero to make the matrix diagonalizable; b must be nonzero to make the matrix not diagonal. Other solutions are $\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$.